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Quantum Deformation of the Two-Dimensional Hydrogen Atom in a Magnetic Field

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The deformed Schrödinger equation for the two-dimensional hydrogen atom in a homogeneous magnetic field is obtained. It is found that the deformed potential belongs to a new set of quasi-exactly solvable potentials.

The quantum deformation of Lie algebras, also called Lie groups (Faddeev, 1984; Drinfeld, 1986; Lukiersky *et al.*, 1991, 1993; Bacry, 1993; Chaichian and Kulish, 1990), has attracted much recent attention. Quantum groups play an important role in conformal field theory (Alvarez Goume *et al.*, 1990), statistical mechanics, inverse scattering theory (Kulish and Sklyanin, 1982), the Yang–Baxter equation (Degasperis and Shabat, 1994) geometrical quantization, etc. In this approach, some authors have realized the $SU_q(2)$ algebra using deformed harmonic oscillator creation and annihilation operators (Arik and Coon, 1976; Biedenharn, 1989; Mcfarlane, 1989). Others have used the $SU_q(2)$ as well as the deformed oscillator structure to determine the effect of deformation on physical observables (Dayi and Duru, 1995; Daskaloyannis and Ypsilartis, 1992; Biedenharn *et al.*, 1993; Ting and Li, 1992; Roy and Roychoudhury, 1995a; Day *et al.*, 1994).

In this paper we show how the exact solution of the deformed wave equation can be obtained for the potential

$$V(r) = -\frac{Z}{r} + \lambda r^2 + \frac{m^2 - 1/4}{2r^2}$$

using the technique of partial algebraization. The potential given above represents a model of the two-dimensional hydrogen atom in a magnetic field,

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and belongs to the class of the so-called quasi-exactly solvable potentials (Roy and Roychoudhury, 1994, 1995b; Turbiner and Ushveridze, 1987; Shifman, 1989). Exact solutions for the more general case, *viz.*, $V(r) = -Z/r + 2gr + \lambda r^2 + l(l+1)/2r^2$, were given by Roychoudhury and Varshni (1988). More recently, Taut (1995) gave analytical solutions for the above potential for the case $g = 0$.

We shall use the finite-dimensional representation of $SU(2)$ and discuss the formalism of $SU_q(2)$ up to $j = 3/2$. Since calculations lose their simplicity and straightforwardness with increasing values of j, and $j = 1$ is the simplest and most elegant example of an exactly deformed Schrödinger equation, we shall discuss in detail the above formalism for $j = 1$. The deformed potential looks completely different from the original one, and has shifted Coulomb potential-like terms, e.g., $1/[1 + (\beta/\alpha)\rho]$ and $1/[1 + (\beta/\alpha)\rho]^2$. We obtain the eigenvalues and eigenfunctions to $O(\tau^2)$ (where $\tau = \ln q$, q being the deformation parameter) and see that when $q \to 1$, our results agree with the undeformed results of Taut (1995).

Now we study the deformation of the Schrödinger problem with potential

$$V(r) = -\frac{Z}{r} + \lambda r^2 + \frac{m^2 - 1/4}{2r^2} \tag{1}$$

The radial wave function $u(r)$ satisfies the two-dimensional radial Schrödinger equation

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{m^2 - 1/4}{2r^2} + \frac{1}{2}w^2r^2 - \frac{Z}{r}\right]u(r) = (E - mw)u(r) \tag{2}$$

where m is the angular momentum, w is the Larmor frequency, and

$$\psi(r) = \frac{u(r)}{\sqrt{r}} \tag{3}$$

For more details see Taut (1995).

In terms of the rescaled variable

$$\rho = \sqrt{w}r \tag{4}$$

equation (2) reduces to

$$\left[-\frac{1}{2}\frac{d^2}{d\rho^2} + V(\rho)\right]u(\rho) = eu(\rho) \tag{5}$$

where

$$V(\rho) = \frac{m^2 - 1/4}{2\rho^2} + \frac{\rho^2}{2} - \frac{Z/\sqrt{w}}{\rho} \tag{6}$$

$$e = \frac{E}{w} - m$$

We now follow the standard method of partial algebraization based on $SU(2)$ algebra. We follow the notations of Taut (1995).

We take the gauge function to be

$$W(\rho) = \rho - \frac{|m| + 1/2}{\rho} \tag{7}$$

so that

$$\begin{aligned} u(\rho) &= \exp(- \textstyle\int w(\rho)\, d\rho] \\ &= e^{-\rho^2/2}\, \rho^{|m|+1/2} \phi(\rho) \end{aligned} \tag{8}$$

and the gauge-transformed Hamiltonian reads

$$H_G = -\frac{1}{2}\frac{d^2}{d\rho^2} + \left\{\rho - \frac{|m| + 1/2}{\rho}\right\}\frac{d}{d\rho} + |m| + 1 - \frac{Z/\sqrt{w}}{\rho} \tag{9}$$

Defining

$$\Omega_G\phi \equiv \rho(H_G - e)\phi \tag{10}$$

we get

$$\Omega_G\phi = \left\{-\frac{1}{2}\rho\frac{d^2}{d\rho^2} + [\rho^2 - (|m| + 1/2)]\frac{d}{d\rho} - \frac{Z}{\sqrt{w}} + \rho(|m| + 1 - e)\right\}\phi \tag{11}$$

In terms of the finite-dimensional representation of the $SU(2)$ group generators, Ω_G can be written as

$$\Omega_G = AT^{0^2} + DT^-T^0 + FT^-T^+ + GT^+ + H_eT^- + IT^0 \tag{12}$$

where $T^0, T^{\pm}$ are the generators of $SU(2)$ algebra which satisfy the following commutation rules:

$$\begin{aligned} [T^+, T^-] &= 2T^0 \\ [T^{\pm}, T^0] &= T^{\pm} \end{aligned} \tag{13}$$

In terms of differential operators, $T^{\pm}$, T^0 may be given by

$$
\begin{aligned}
T^{+} &= 2j\xi - \xi^2 \frac{d}{d\xi} \\
T^{0} &= -j + \xi \frac{d}{d\xi} \\
T^{-} &= \frac{d}{d\xi}
\end{aligned}
\tag{14}
$$

With the help of (14), Ω_G then takes the form

$$
\begin{aligned}
\Omega_G = {}& \{(A - F)\rho^2 + D\rho\} \frac{d^2}{d\rho^2} \\
& + \{[(1 - 2j)A + 2(j - 1)F + I]\rho - G\rho^2 + H_e + D(1 - j)\} \frac{d}{d\rho} \\
& + \{2jG\rho + Aj^2 + 2jF - jI\}
\end{aligned}
\tag{15}
$$

Comparing (11) and (15), we obtain

$$
\begin{aligned}
A &= -\frac{Z}{\sqrt{w}} \frac{1}{j(j + 1)} = F = I \\
D &= -1/2 \\
G &= -1 \\
H_e &= -|m| - j/2 \\
E &= w\{1 + 2j + m + |m|\} \\
e &= 1 + 2j + |m|
\end{aligned}
\tag{16}
$$

To obtain the deformed Schrödinger equation, we keep the coefficients A, D, G, H_e, F, and I unchanged, but replace $T^{\pm}$, T^0 by $T_q^{\pm}$, T_q^0 in (12), where $T_q^{\pm}$, T_q^0 satisfy the commutation relations

$$
\begin{aligned}
[T_q^{+}, T_q^{-}] &= [2T_q^0] \\
[T_q^{\pm}, T_q^0] &= T_q^{\pm}
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
[n] &= \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\sinh \tau n}{\sin \tau}, \\
\tau &= \ln q
\end{aligned}
\tag{18}
$$

q is the deformation parameter; it is a real number.

Case 1. First we consider the case $j = 1/2$. Then $T^{\pm}$, T^0 are given by

$$T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{19}$$
$$T^0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

It is found that in this case

$$\frac{\sinh 2\tau T^0}{\sinh \tau} = 2T^0$$

Hence, no deformation occurs.

So

$$T_q^{\pm} = T^{\pm}, \qquad T_q^0 = T^0 \tag{20}$$

and Ω_G^q $(= \Omega_G)$ reduces to

$$\Omega_G^q = \Omega_G = \begin{pmatrix} -Z/\sqrt{w} & -1 \\ -|m| - \frac{1}{2} & -Z/\sqrt{w} \end{pmatrix} \tag{21}$$

The eigenvalue equation

$$(H_G^q - e_q)\phi = 0$$

implies

$$\det \Omega_G^q = 0$$

which leads to

$$w = \frac{Z^2}{|m| + 1/2} \tag{22}$$
$$E = w(m + |m| + 2)$$

This result agrees with that of the analytical approach to the undeformed problem in Taut (1995).

Case 2. $j = 1$. The $SU(2)$ generators in this case are given by

$$T^+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{23}$$

$$T^0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

It is seen that

$$\frac{\sinh 2\tau T^0}{\sinh \tau} = [2]T^0$$

so that we may take the following realization of $T_q^{\pm}$, T_q^0:

$$T_q^{\pm} = \alpha T^{\pm}, \qquad T_q^0 = T^0 \tag{24}$$

where

$$\alpha = ([2]/2)^{1/2} \tag{25}$$

Thus the deformed equation for Ω_G takes the form

$$\Omega_G^q = AT^{0^2} + \alpha DT^- T^0 + \alpha^2 FT^- T^+ + \alpha GT^+ + \alpha H_e T^- + IT^0 \tag{26}$$

Eigenvalues are determined by the condition

$$\det \Omega_G^q = 0$$

which gives

$$w = \frac{\alpha^2 Z^2}{(2|m| + 1) + 2\alpha^2(|m| + 1)} \tag{27}$$

$$E = w\{|m| + m + 3\}$$

Thus w and E now depend on the deformation parameter α.

For small deformation,

$$\alpha = \left(\frac{[2]}{2}\right)^{1/2} \simeq 1 + \frac{\tau^2}{4} \tag{28}$$

so that, up to $O(\tau^2)$,

$$w = \frac{Z^2}{4|m| + 3}\left\{1 + \left(\frac{\tau^2}{2}\right)\frac{2|m| + 1}{4|m| + 3}\right\} \tag{29}$$

Thus our result agrees with that of Taut (1995) in the absence of any deformation.

In terms of the differential operator, Ω_G^q can be written as

$$\begin{aligned}\Omega_G^q = &\left\{-\frac{Z}{2\sqrt{w}(1 - \alpha^2)}\rho^2 - \frac{\alpha}{2}\rho\right\}\frac{d^2}{d\rho^2}\\ &+ \left\{(1 - \alpha^2)(j - 1)\frac{Z}{\sqrt{w}}\rho + \alpha\rho^2 - \alpha\left(|m| + \frac{1}{2}\right)\right\}\frac{d}{d\rho}\\ &- \{(j - 1) + 2\alpha^2\}\frac{Z}{2\sqrt{w}} - \alpha\rho(e - |m| - 1)\end{aligned} \tag{30}$$

To get back the Schrödinger equation, we use the reverse transformation in the equation

$$\Omega_G^q\phi_q(\rho) = 0 \tag{31}$$

with

$$\phi_q(\rho) = \rho^{-(|m|+1/2)}\left(1 + \frac{\beta}{\alpha}\rho\right)^{\lambda} e^{(\alpha/\beta)\rho}u_q(\rho) \tag{32}$$

where

$$\begin{aligned}\beta &= \frac{Z}{\sqrt{w}}(1 - \alpha^2)\\ \lambda &= |m| + \frac{1}{2} - \frac{\alpha^2}{\beta^2}\end{aligned} \tag{33}$$

It can be shown that

$$\phi_q(\rho) = 1 + a_1\rho + a_2\rho^2 \tag{34}$$

is a solution of equation (31) provided

$$\begin{aligned}a_1 &= -\frac{\alpha Z}{\sqrt{w}}\frac{1}{|m| + 1/2}\\ a_2 &= \frac{\alpha^2}{|m| + 1/2}\end{aligned} \tag{35}$$

Thus $u_q(\rho)$ satisfies the deformed Schrödinger equation

$$\left\{-\frac{1}{2}\frac{d^2}{d\rho^2} + (V_q(\rho) - e_q)\right\} u_q(\rho) = 0 \tag{36}$$

where the deformed potential $V_q(\rho)$ and the deformed energy e_q are given by

$$\begin{aligned} V_q(\rho) - e_q = {} & \frac{m^2 - 1/4}{2\rho^2} - \frac{1}{\rho}\left\{\frac{\alpha Z}{\sqrt{w}} + \frac{\beta}{\alpha}\left(|m| + \frac{1}{2}\right)^2\right\} \\ & + \frac{1}{1 + (\beta/\alpha)\rho}\left\{\frac{\beta Z}{\sqrt{w}} - (e - |m| - 1) - \frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2}\left(|m| + \frac{1}{2}\right)^2\right\} \\ & + \frac{1}{[1 + (\beta/\alpha)\rho]^2}\left\{\frac{\alpha^2}{\beta^2} + \left(|m| + \frac{1}{2}\right)\left(|m| + \frac{3}{2}\right)\frac{\beta^2}{\alpha^2} - 2(|m| + 1)\right\} \end{aligned} \tag{37}$$

For small values of the deformation parameter, up to $O(\tau^2)$, (37) reduces to

$$\begin{aligned} V_q(\rho) - e_q = {} & \frac{m^2 - 1/4}{2\rho^2} + \frac{\rho^2}{2} - \frac{Z/\sqrt{w}}{\rho} - e \\ & + \left(\frac{\tau^2}{2}\right)\frac{Z}{\sqrt{w}}\left\{\rho^2 - (e + |m| + 1)\rho \right. \\ & \left. + \frac{(|m| + 1/2)^2 - 1/2}{\rho} - \frac{Z}{\sqrt{w}}\right\} \end{aligned} \tag{38}$$

and the wave function assumes the form

$$\begin{aligned} U_q(\rho) = {} & e^{-\rho^2/2} \exp\left\{-\frac{\tau^2}{2}\frac{Z}{\sqrt{w}}\left(\frac{\rho^3}{3} - \left(|m| + \frac{1}{2}\right)\rho\right)\right\} \\ & \times \rho^{|m| + 1/2}\left\{1 - \frac{Z}{|m| + 1/2}\frac{\rho}{\sqrt{w}} + \frac{1}{|m| + 1/2}\rho^2 \right. \\ & \left. + \frac{\tau^2}{|m| + 1/2}\left(\frac{\rho^2}{2} - \frac{Z}{4}\frac{\rho}{\sqrt{w}}\right)\frac{\rho}{\sqrt{w}}\right\} \end{aligned} \tag{39}$$

It can be shown that in the limit $\tau \to 0$, our results agree completely with the undeformed result of Taut (1995).

Case 3. $j = 3/2$. Eigenvalues for higher values of j are less straightforward to calculate. The reason for this is that there is no simple relationship between the deformed and underformed generators of $SU(2)$, unlike the case $j = 1$

[see equation (24)]. Hence $T_q^{\pm}$, T_q^0 cannot be given a simple form in terms of the differential operators, which, in turn, prevents one from obtaining the deformed potential in such cases. However, one can always obtain the deformed energy and the deformed wave function by writing explicitly the matrices for $T_q^{\pm}$, T_q^0. For example, for $j = 3/2$

$$
\begin{aligned}
T_q^+ &= \begin{pmatrix} 0 & \sqrt{[3]} & 0 & 0 \\ 0 & 0 & [2] & 0 \\ 0 & 0 & 0 & \sqrt{[3]} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
T_q^- &= (T_q^+)^+ \\
T_q^0 &= \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}
\end{aligned}
\tag{40}
$$

In terms of these deformed generators, the gauge-transformed equation reads

$$\Omega_G^q = AT_q^{0^2} + DT_q^- T_q^0 + FT_q^- T_q^+ + GT_q^+ + H_e T_q^- + IT_q^0 \tag{41}$$

where A, D, F, G, H_e, and I are given by (16).

Hence the deformed energy eigenvalue is obtained from

$$E_q = \omega_q(|m| + m + 4) \tag{42}$$

where ω_q is determined from the condition

$$\det \Omega_G^q = 0 \tag{43}$$

From (40), (41), and (43) we get, after some algebra,

$$\omega_q = \left(\frac{4Z}{15}\right)^2 \left\{\frac{-a_2 \pm \sqrt{a_2^2 - 4a_1a_3}}{2a_1}\right\} \tag{44}$$

where

$$
\begin{aligned}
a_1 &= \left(|m| + \frac{1}{2}\right)\left(|m| + \frac{3}{2}\right)[3]^2 \\
a_2 &= -\left(\frac{3}{4} + [3]\right)\left\{\frac{15}{4}[2]^2(|m| + 1) - \frac{15}{4}[3]\left(|m| + \frac{1}{2}\right)\right. \\
&\quad \left. + \left([2]^2 - \frac{1}{4}\right)[3]\left(|m| + \frac{3}{2}\right)\right\}
\end{aligned}
\tag{45}
$$

$$a_3 = -\frac{15}{4}\left([2]^2 - \frac{1}{4}\right)\left(\frac{3}{4} + [3]\right)^2$$

Thus, analogous to the undeformed case, there are two solutions for $j = 3/2$, even in the deformed version. For small deformation [using (18)]

$$a_1 = a_{10}\left(1 - \frac{24}{9}\tau^2\right)$$
$$a_2 = a_{20} + \frac{15}{4}\tau^2(43|m| + 76) \tag{46}$$
$$a_3 = a_{30}\left(1 - \frac{48}{15}\tau^2\right)$$

where a_{10}, a_{20} and a_{30} stand for a_1, a_2, and a_3, respectively, in the absence of deformation, given by

$$a_{10} = 9\left(|m| + \frac{1}{2}\right)\left(|m + \frac{3}{2}\right)$$
$$a_{20} = -\left(\frac{15}{4}\right)^2(4|m| + 7) \tag{47}$$
$$a_{30} = -\left(\frac{15}{4}\right)^4$$

so that

$$\omega_0 = \left(\frac{4Z}{15}\right)^2\left\{\frac{-a_{20} \pm \sqrt{a_{20}^2 - 4a_{10}a_{30}}}{2a_{10}}\right\}$$
$$= Z^2\left\{\frac{(4|m| + 7) \pm 2\sqrt{13|m|^2 + 32|m| + 19}}{18(|m| + 1/2)(|m| + 3/2)}\right\} \tag{48}$$

To the best of our knowledge even this undeformed result has not been given in the literature.

To conclude, we have shown that there is no deformation of the 2D hydrogen atom in a magnetic field in the case $j = 1/2$, and we get back the results of Taut (1995). For $j = 1$, the deformation gives a new set of quasi-exactly solvable potentials. To the lowest order, i.e., $O(\tau^2)$, this potential gives linear and cubic terms in ρ (which is directly related to τ), apart from the 2D hydrogen atom potential. Also the results of Taut (1995) are reproduced in the limit $\tau \to 0$, *i.e.*, when the deformation parameter vanishes.

For higher j values, calculations are less straightforward due to the absence of a simple relationship between $T_q^{\pm}$, T_q^0 and $T^{\pm}$, T^0 and it is not easy to see the nature of the deformed potential. However, the deformed energy can be always obtained by writing explicitly the matrices for $T_q^{\pm}$, T_q^0 as we have shown for the case $j = 3/2$.

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